NASH-BARGAINED HOUSEHOLD DECISIONS: TOWARD A GENERALIZATION OF THE THEORY OF DEMAND*

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1. INTRODUCTION

The theoretical underpinnings of much empirical work in economics is, either implicitly or explicitly, individual constrained utility maximization and the resulting system of demand equations. Often times, however, the behavior being modeled is the outcome of group, not individual, decisions. Recent studies of the allocation of the time of individual family members between home and the market are a case in point: Ashenfelter and Heckman [1974], Heckman [1974, 1977], and Wales and Woodland [1976, 1977]. In this paper we present alternative theoretical underpinnings for such studies, a Nash-bargained system of demand equations and the comparative statics of the associated demand system. Attention is focused on contrasting the empirical implications of the Nash bargaining model with those of neoclassical individual utility maximization.

To be concrete, we work with a two-person household, a "married couple," and their joint allocation of money-income and time. The model, however, is applicable to outcomes of any decisions that can be structured as a constrained, static, two-person, non-zero-sum game. The model can also be generalized to more than two players. Although we analyze only the outcome of a two-person cooperative game with a Nash [1953] solution, our goal is broader: to illuminate the general characteristics of the empirically observable differences between bargained and individual decision making. Our approach contrasts with those of Becker [e.g., 1973] and Samuelson [1956]. They place sufficient a priori structure (in the form of "full caring" and a "marginal-ethical-worth" rule, respectively) on "family" decisions so that the outcomes are empirically indistinguishable from those of constrained individual utility maximization. We refer to these and other approaches which explicitly or implicitly reduce family decisions to individual decisions as "neoclassical."

The empirical distinctions between Nash-bargained and neoclassical household decisions are highlighted in the explicit expressions we derive for Nash generalizations of the neoclassical Slutsky equation, of neoclassical substitution symmetry, and of Engel aggregation. The Nash generalization of Engel aggregation arises

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2 Many of the comparative static results were presented in Horney [1977]. If nonwage income were not an argument of the threat points but prices remained so, then our model would be a special case of (conditional) price dependent preferences as surveyed by Pollak [1977].
because, in contrast to a neoclassical household, for a Nash household male and female nonwage incomes are distinct arguments of each demand equation. Since the usual neoclassical comparative static results are nested in their Nash generalizations, we propose empirical tests that the Nash demand system collapses to a neoclassical one. In a companion paper, Horney and McElroy [1980], using a three equation linear expenditure system on married, working couples from the 1967 National Longitudinal Survey of Mature Women, we find evidence to reject the proposition that the Nash comparative statics collapse to the neoclassical ones.

The work of Manser and Brown [1978, 1980] and Brown and Manser [1977, 1978] is both parallel and complementary to ours. Using a Nash bargaining specification similar to ours, they demonstrate the existence of household demand functions. Their work is broader than ours in that they analyze bargaining rules other than the Nash type, and look at the empirical distinctions among these as well as between these and the neoclassical demand system. In addition, Manser and Brown analyze the demand for marriage. In contrast, our work is more narrowly focused on the Nash household model and a detailed and explicit analysis of the comparative statics, empirical implications, and specification of tests of the hypothesis that the Nash model collapses to the neoclassical demand model. In related work, Clemhout and Wan [1977] use a Nash solution to a differential game to determine income shares which, in turn, dictate the allocation of household expenditures through a Lindahl equilibrium. Thus, changes in relative income shares explain changes in consumption and work patterns.

Section 2 presents the Nash model and the associated complete system of demand equations. Section 3 presents the Nash analogue to neoclassical indifference curves and their relationship to prices and incomes. Section 4 derives the comparative statics of the Nash model, including the Nash generalizations of the Slutsky equation, of substitution symmetry and of Engel aggregation. In this section, the usual neoclassical comparative static results are shown to be nested in those of the Nash model; based on this nesting, empirical tests are proposed. In Section 5 a graphical analysis is used to summarize the relationship between the Nash model and neoclassical model. Section 6 contains brief concluding remarks.

2. THE NASH HOUSEHOLD OBJECTIVE FUNCTION AND DEMAND SYSTEM

Assume that a household consists of two people, $m$ and $f$. Their objects of choice are $x = (x_0, x_1, x_2, x_3, x_4)'$ with given market prices $p = (p_0, p_1, p_2, p_3, p_4)'$. Here $x_1$ is a market good consumed by the husband; $x_2$ is a market good consumed by the wife; $x_3$ is the quantity of "leisure" of the husband; $x_4$ is the quantity of "leisure" of the wife; and $x_0$ is a "household good." The household goods are defined as a pure public good within the household: consumption of $x_0$ by one

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2 It is straightforward to generalize the theory that follows by taking each $x_i$ to be a vector. Few, if any, insights are lost by ignoring this complication.
household member does not reduce the amount available to the other. "Leisure" time includes all time not spent at market work. It will prove convenient to classify goods and prices as those of direct interest to $m$,

$$x_m = (x_0, x_1, x_3)' \quad \text{at} \quad p_m = (p_0, p_1, p_3)'$$

and those of direct interest to $f$,

$$x_f = (x_0, x_2, x_4)', \quad \text{at} \quad p_f = (p_0, p_2, p_4)'$$

We assume that if $m$ and $f$ were not married, each would behave as if to maximize a twice continuously differentiable, nondecreasing, quasiconcave utility function, subject to that individual's budget restraint,

1. $$I_m + p_3(T - x_3) = p_0x_{0m} + p_1x_1, \quad \text{for} \quad m$$
2. $$I_f + p_4(T - x_4) = p_0x_{0f} + p_2x_2, \quad \text{for} \quad f.$$

Here $x_{0k}$ is the amount of the "household" good each consumes; $I_k$ is nonwage income for $k=m, f$; and $T$ is total time to be allocated between market and nonmarket pursuits. Consequently, each person has a well defined continuous, strictly quasiconvex indirect utility function giving the maximum attainable utility level as a function of prices and nonwage income:

3. $$V^m_0 = V_0^m(p_m, I_m) \quad \text{and} \quad V^f_0 = V_0^f(p_f, I_f).$$

Duality insures that the partial derivatives of the indirect utilities have the indicated signs: \footnote{For convenience, we define as zero the partial of $V^k_\delta$ with respect to a variable which is not an argument of $V^k_\delta$, e.g., $\frac{\partial V^k_\delta}{\partial p_3} = 0$.}

\begin{table}
\centering
\begin{tabular}{c|cccccccc}
\hline
 & $\frac{\partial V^m_\delta}{\partial p_0}$ & $\frac{\partial V^m_\delta}{\partial p_1}$ & $\frac{\partial V^m_\delta}{\partial p_2}$ & $\frac{\partial V^m_\delta}{\partial p_3}$ & $\frac{\partial V^m_\delta}{\partial I_m}$ & $\frac{\partial V^m_\delta}{\partial I_f}$ \\
\hline
$k=m$ & - & - & 0 & + & 0 & + \\
$k=f$ & - & 0 & - & 0 & + & + \\
\hline
\end{tabular}
\end{table}

For a couple, however, the utility of each is assumed to depend not only on own goods, own leisure, and the household good, but also upon the nonmarket time and the consumption of the spouse as well. Thus their individual utility functions are given by

4. $$U^k = U^k(x), \quad \text{for} \quad k = m, f.$$

Hence, the gain from being married as opposed to single, assumed non-negative for existing marriages, \footnote{See Manser and Brown [1980] for an elaboration of this point.} is

5. $$U^k(x) - V^k_0(p^*_k, I_k) \quad \text{for} \quad k = m, f.$$

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In this model, a married couple is not necessarily distinguished from two single individuals on legal grounds, but rather on the basis of pooling resources and allocating them jointly. Thus, total expenditures on the household good, own goods and leisures equals “full income,” or

\[ p_0 x_0 + p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 = (p_3 + p_4)T + I_m + I_f. \]

We assume that bargaining over the allocation of \( x \) achieves the Nash solution to a two-person, nonzero-sum game. As is well known, this solution is characterized by invariance with respect to linear transformations of individual utility functions, Pareto efficiency of the allocation of resources, independence of irrelevant alternatives and symmetry (in terms of utilities) with respect to the roles of the players. We will not debate here the merits of this particular set of criteria, but refer the reader to Manser and Brown [1980]. However, we would emphasize that we think the most serious deficiency of the Nash solution as applied to this household game may lie not in the assumption of the independence of irrelevant alternatives or symmetry but rather in the inadmissibility of interpersonal comparisons of utility.

Thus, we assume the couple chooses \( x \) to maximize, subject to (6), what we call the “utility-gain product function,” a special case of the Nash product function,

\[ N = [U^n(x) - V^n_0(p^n_m, I_m; z_m)] [U^f(x) - V^f_0(p^f_f, I_f; z_f)]. \]

Each term in brackets is the gain from marriage over the next best alternative; \( V^k_0 \) is reinterpreted as the threat point of the \( k \)-th individual and represents the best he could expect to do if he were to withdraw from the household. The threat points may shift because the opportunities outside of the marriage change. We define the \( z_k \)’s as the relevant shift parameters. For example, \( z_k \) might be a search parameter such as the ratio of males to females in the relevant marriage market. Setting the first partials of the appropriate Lagrangian function equal to zero yields the necessary conditions for a maximum. These are unique up to linear transformations of the \((U^k-V^k)\):

\[ N_i \equiv U^n_i(U^f - V^f_0) + (U^n - V^n_0)U^f_i = \lambda p_i, \quad i = 0, 1, 2, 3, 4. \]

6 The class of admissible utility functions is restricted. For example, let \( m \)'s utility function as a single person be \( U^n(x) \) with \( x_1 \) and \( x_3 \) constrained to be zero. Then clearly certain functions such as \( U^n = x_1 x_2 x_3 x_4 x_5 \equiv 0 \) for all \((x_0, x_1, x_3)\) are inadmissible for then \( V^n_0 \equiv 0 \) for all \((p^n_m, I_m)\).

7 Luce and Raiffa [1957, pp. 130-132] briefly discuss the possible advantages of admitting interpersonal comparisons at the expense of independence. In the broader context of collective choice Sen [1977, pp. 1550-1552] gives references to rules incorporating both independent and interpersonal comparisons, neither one, and independence without interpersonal comparisons. He then gives the first discussion of the remaining combination: rules using interpersonal comparisons without independence.

8 This maximum is not necessarily associated with being single, but instead with attaining the highest level of utility among all alternatives to the marriage.

9 In an interesting elaboration of this point, Manser and Brown [1980] take the threat point to be the expected utility from continued search for a partner.
(9) \[-N_i \equiv p_0 x_0 + p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 - T(p_3 + p_4) - I_m - I_f = 0,\]

where \(N_i = \frac{\partial N}{\partial x_i}\), \(U_k = \frac{\partial U_k}{\partial x_i}\) for \(k = m, f\) and \(i = 0, 1, \ldots, 4\); \(N_\lambda = \frac{\partial N}{\partial \lambda}\), and \(\lambda\) is the Lagrange multiplier. Solving (8) and (9) (locally) for the optimal \(x\) as a function of prices and nonwage incomes\(^{10}\) yields the demand system and the associated optimal value of the Lagrange multiplier:

(10) \[x_i = h_i(p', I_m, I_f), \quad i = 0, 1, 2, 3, 4,\]

(11) \[\lambda = h_\lambda(p', I_m, I_f).\]

The multiplier, \(\lambda\), is the increase in the utility-gain product due to a unit increase in full income, evaluated at the optimal \(x\). Since \(V^*_a\) and \(V'_f\) are homogeneous of degree zero in prices and income, (8) guarantees that the demand functions (10) are homogeneous of degree zero in prices and incomes.

3. ISO-GAIN PRODUCT CURVES AND THE FAMILY RATE OF SUBSTITUTION

In the comparative statics of the neoclassical utility model, changes in prices and income change the optimal bundle only via twists and shifts in the budget restraint while the indifference map remains fixed. In the comparative statics of a bargaining model there is an additional complication: changes in prices and nonwage incomes not only twist and shift the budget restraint but also change the objective function itself. These changes in the objective function are easily characterized as twists and shifts in the "iso-gain product curves" (the bargaining analogue of neoclassical indifference curves): the twists are changes in the "family rate of substitution" (the bargaining analogue of neoclassical marginal rates of substitution); the shifts are changes in the level of the value attached to an iso-gain product curve.

3.1. Definition of IGPC. Define an iso-gain product curve (IGPC) as the locus of bundles, \(x\), for which the utility-gain product function is constant. For given prices and incomes, there is a family of iso-gain product curves which share the following properties with neoclassical indifference curves: (a) two curves cannot intersect; (b) an IGPC with a higher value lies everywhere to the northeast of any IGPC with a lower value; (c) there is an IGPC passing through every point in the nonnegative orthant of the space in which the \(x\)'s lie.

3.2. Changes in \(FRS_{ij}\). The "family" rate of substitution of \(x_i\) for \(x_j\) is defined as minus the slope of the IGPC at \(x\), or

(12) \[FRS_{ij} = -\frac{dx_j}{dx_i} \bigg|_{N} = \frac{N_i}{N_j}, \text{ and } dx_k = 0 \text{ for } k \neq i, k \neq j, i \neq j, \]

\[k, i, j = 0, 1, 2, \ldots, 4.\]

\(^{10}\) We assume that the \(U^*\) and \(V'_f\) are such that \(N\) is a quasiconcave function and that the implicit function theorem holds.
At $x$, $FRS_{ij}$ depends upon the arguments of the threat points: $p$, $l_m$, $l_f$, $x_m$ and $x_f$. Let $z$ stand for any particular one of these arguments. Then

$$\frac{\partial FRS_{ij}}{\partial z} = U_m^i \frac{U_f^j}{N_j^2} \left( \frac{U_f^i}{U_f^j} - \frac{U_m^i}{U_m^j} \right) \left( \frac{\partial V_m^0}{\partial z} (U^m - V_m) - \frac{\partial V_m^m}{\partial z} (U_f - V_f) \right)$$

$$= \frac{U_m^i U_f^j}{N_j^2} (AMRS_{ij}) [W], \text{ say, for } i,j = 0, 1, 2, 3, 4 \text{ and } i \neq j.$$

Since $U_m^i U_f^j / N_j^2 > 0$, the sign of $\frac{\partial FRS_{ij}}{\partial z}$ depends on those of $AMRS_{ij}$ and of $W$.

If $z$ is any one of the following, $p_1, p_2, p_3, p_4, l_m$, or $l_f$, then the sign of $W$ is determined. For in these cases, one of the terms $\frac{\partial V_m^0}{\partial \sigma_m}$ and $\frac{\partial V_m^m}{\partial \sigma_m}$ will have a known sign (see Table 1 above), and the other will be zero. Finally if $z = x_m = x_f$ but $\frac{\partial V_m^0}{\partial \sigma_m}$ and $\frac{\partial V_m^m}{\partial \sigma_m}$ have opposite signs, then once again the sign of $W$ is determined.

For example, let $z = x_m = x_f$ be the ratio of the number of males to the number of females in the relevant marriage market. Then $\frac{\partial V_m^0}{\partial \sigma_m} < 0$, $\frac{\partial V_m^m}{\partial \sigma_m} > 0$ and $W$ is positive. Hence the sign of $W$ is determined in all but two cases: when $z$ is $p_0$ and when $z$ is $x_m = x_f$ and sign $\frac{\partial V_m^0}{\partial \sigma_m} =$ sign $\frac{\partial V_m^m}{\partial \sigma_f}$.

$AMRS_{ij}$ is the difference in the spouses' individual marginal rates of substitution at $x$. If $AMRS_{ij} > 0$, then she places a higher relative value on $x_i$ (in terms of $x_j$) than he does.

**Definition.** If $x_i$ and $x_j$, with $i \neq j$, are both of direct interest to her (him), then he (she) says to be **nonjudgmental** with respect to $x_i$ and $x_j$ at $x$ if at $x$,

$$U_m^i \frac{U_f^j}{U_m^j} = U_m^j \frac{U_f^i}{U_m^i}.$$  \hspace{1cm} (14)

If the relationship (14) is changed to inequality, say $>$, then he is **paternalistic** and pushing $x_i$ at the expense of $x_j$ (she is **maternalistic** and pushing $x_j$ at the expense of $x_i$); or, he (she) views $x_i$ ($x_j$) as a **merit good**.

**Definition.** If $x_i$ is of direct interest to him (her) and $x_j$, only of indirect interest to him (her), then they are **selfish** at $x$.

$$U_f^i \frac{U_m^j}{U_f^j} < U_m^i \frac{U_f^j}{U_m^j} \quad \left( \text{if} \quad \frac{U_f^i}{U_f^j} > \frac{U_m^i}{U_m^j} \right).$$  \hspace{1cm} (15)

They are **altruistic** if the opposite relationship holds, and **neutral** in the case of equality.

\textsuperscript{11} Since $\frac{\partial FRS_{ij}}{\partial z} = N_j \frac{\delta N_i}{\delta z} - N_i \frac{\delta N_j}{\delta z}$, (for $i,j = 0, 1, 2, 3, 4$ and $i \neq j$), substituting in (8) as well as $\frac{\delta N_i}{\partial z}$ \left( \frac{\partial V_m^0}{\partial z} U_f + \frac{\partial V_m^m}{\partial z} U_m \right)$ gives (13).
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For the many cases where we can sign $W$ and make an assumption with respect to $\Delta MRS_{ij}$ (e.g., selfishness), we can sign $\frac{\partial FRS_{ij}}{\partial z}$ and thereby know, for a region of the space in which the x's lie, the direction of the twist in the IGPC map. For example, let $(x_i, x_j) = (x_1, x_2)$, his and her consumption goods. Assume selfishness at $x$, i.e., $\Delta MRS_{12} < 0$. Then $W < 0$ and $\frac{\partial FRS_{12}}{\partial I_m} > 0$ in a neighborhood of $x$. In words, selfishness implies that an increase in his nonwage income changes the family decision function to reflect a relatively higher value placed on his consumption good.

3.3. Changes in the IGPC level. Changes in $z$ also change the value attached to the IGPC through any point, $x$. In general

\[
\frac{\partial N}{\partial z} = -(U^m - V_0^m) \frac{\partial V_0}{\partial z} - (U^f - V_0^f) \frac{\partial V_0}{\partial z}.
\]

From Table 1 above, we can sign $\frac{\partial N}{\partial z}$ when $z$ is any price, including $p_0$, and when $z$ is $I_m$ or $I_f$. In Section 5 below, we use these changes in the numerical value assigned to an IGPC through $x$ for expositing the effect of income-compensated price changes. As the reader may suspect, however, such changes do wash out of the empirical implications.

4. COMPARATIVE STATICS OF THE NASH MODEL: A GENERALIZATION OF THE NEOCLASSICAL MODEL

In this section we present the uncompensated effects of price changes, the compensated effects of price changes, and the effects of income changes on both the optimal bundle ($x$) and the Lagrange multiplier ($\lambda$). These effects lead naturally to a Nash generalization of Engel aggregation, of the Slutsky equation, and of substitution symmetry. The derivations of many of the relationships in this section are relegated to the Appendix.

4.1. Notation. The vector of excess demands for $x$ is

\[
q = x' - (0, 0, 0, T, T)' = (x_0, x_1, x_2, x_3 - T, x_4 - T)'.
\]

The last two elements of $q$ are the negatives of his and her excess supply of hours of market work. The $j$-th column of the following $X$-matrices and the $j$-th element of the $\lambda$-vectors contain the partial derivatives of the optimal bundle ($x$) and of $\lambda$ with respect to the $j$-th price:

\[
X_p = \left[ \frac{\partial x_i}{\partial p_j} \right]_{(5 \times 5)} , \quad \lambda_p = \left( \frac{\partial \lambda}{\partial p_0}, \frac{\partial \lambda}{\partial p_1}, \ldots, \frac{\partial \lambda}{\partial p_4} \right)',
\]

\[
X^*_p = \left[ \frac{\partial x^*_i}{\partial p_j} \right]_{(5 \times 5)} , \quad \lambda^*_p = \left( \frac{\partial \lambda^*}{\partial p_0}, \frac{\partial \lambda^*}{\partial p_1}, \ldots, \frac{\partial \lambda^*}{\partial p_4} \right)'.
\]

Here an asterisk indicates that $N$ is being held constant. The partials of the
optimal \( x \) and \( \lambda \) with respect to \( I_m \) and \( I_f \) are in

\[
X_I = [X_{I_m}, X_{I_f}] = \begin{bmatrix}
\frac{\partial x}{\partial I_m}, \frac{\partial x}{\partial I_f}
\end{bmatrix}, \quad \lambda_I = \begin{bmatrix}
\frac{\partial \lambda}{\partial I_m}, \frac{\partial \lambda}{\partial I_f}
\end{bmatrix}.
\]

The effect of prices and incomes on \( m \)'s and \( f \)'s threat points are captured in

\[
V_p = \begin{bmatrix}
\frac{\partial V_m}{\partial p_0} & \frac{\partial V_m}{\partial p_1} & 0 & \frac{\partial V_m}{\partial I_m} & 0 \\
\frac{\partial V_f}{\partial p_0} & 0 & \frac{\partial V_f}{\partial p_2} & \frac{\partial V_f}{\partial p_3} & \frac{\partial V_f}{\partial I_f}
\end{bmatrix}, \quad V_I = \begin{bmatrix}
\frac{\partial V_m}{\partial I_m} & 0 \\
0 & \frac{\partial V_f}{\partial I_f}
\end{bmatrix}.
\]

The effects on the threat point of one spouse will usually be weighted by either the marginal utilities of the opposite spouse which appear in the columns of

\[
D = \begin{bmatrix}
U_f^0 & U_f^1 & U_f^2 & U_f^3 & U_f^4 \\
U_m^0 & U_m^1 & U_m^2 & U_m^3 & U_m^4
\end{bmatrix},
\]

or by the "gain from marriage" of the opposite spouse which appears in

\[
g = (U_f - V_f, U_m - V_m)' > (0, 0)'.
\]

Finally, \( b, c, \) and \( B \) are the \((5 \times 1), (1 \times 1), \) and symmetric \((5 \times 5)\) submatrices of the inverse of the appropriate bordered Hessian for the optimization problem; \( I_5 \) is the identity matrix of size \( 5 \); and \( \epsilon \) is a \( 2 \times 1 \) vector of ones.

### 4.2. Comparative Statics of the Nash Model

The uncompensated effects of changing the \( j \)-th price — other prices and incomes constant — on the optimal \((x, \lambda)\) are the \( j \)-th column of

\[
X_p = \begin{bmatrix}
B(\lambda I_5 + DV_p) - bq' \\
-\lambda_p'
\end{bmatrix},
\]

which is obtained by partial differentiation of (8) and (9) with respect to each \( p_j \) and then solving for \( X_p, \lambda_p' \). (Details of the derivation are contained in the Appendix.)

The income compensated effects of changing the \( j \)-th price — holding other prices and the level of \( N \) constant — are the \( j \)-th column of

\[
X_p^* = \begin{bmatrix}
B(\lambda I_5 + DV_p) + \lambda^{-1}bg'V_p \\
-\lambda_p^*
\end{bmatrix},
\]

which is obtained by partial differentiation of (7) and (8) with respect to each \( p_j \) and then solving for \( X_p^*, \lambda_p^* \). As noted in Section 3.3 above, the level of \( N \) is of no intrinsic interest. Equation (18) is, however, essential to understanding the Nash generalization of the Slutsky equation and the like.

Equal changes in male and female incomes have different effects on the Nash
objective function (7) and, consequently, on the optimal bundle. The effects of changing the k-th spouse’s income $I_k$ — all prices and the other spouse’s income $I_k$ constant — on the optimal $(x, \lambda)$ are the k-th column of

$$
\begin{bmatrix}
X_l \\
-\lambda_l
\end{bmatrix} = \begin{bmatrix}
BDV_l + bt' \\
 b'DV_l - c^{-1}t'
\end{bmatrix}
$$

which is obtained via partial differentiation of (8) and (9) with respect to $I_m$ and $I_f$ — and then solving for $X_l$, $\lambda_l$.

To emphasize the separate impacts of $I_m$ and $I_f$ on the optimal $x$, write the first line of (19) as,

$$
X_l = [X_{lm}, X_{lf}] = \begin{bmatrix}
BD_{em} \frac{\partial V^e_m}{\partial I_m} \\
BD_{ef} \frac{\partial V^f_o}{\partial I_f}
\end{bmatrix} + [b, b],
$$

where $e_m = (1, 0)'$ and $e_f = (0, 1)'$. These separate impacts also show up in the Nash generalization of Engel aggregation,$^{12}$

$$
p'X_l = \epsilon', \quad \text{or both } p'X_{lm} = 1 \text{ and } p'X_{lf} = 1.
$$

The Nash generalization of the Slutsky equation is obtained by substitution of (18) and (19)$^3$ into (17) to get

$$
\begin{bmatrix}
X_p \\
\lambda_p
\end{bmatrix} = \begin{bmatrix}
(X^*_p - \lambda^{-1}bq'V_p) - \frac{1}{2} (X_l - BDV_l)\epsilon q' \\
(\lambda^*_p - (c\lambda)^{-1}q'V_p) + \frac{1}{2} (\lambda_l - bDV_l)\epsilon q'
\end{bmatrix}.
$$

Further substitutions yield the Nash generalization of substitution symmetry,$^{14}$

$$
\left(X_p + \frac{1}{2} X_l\epsilon q'\right)G^{-1} = (G^{-1})\left(X_p + \frac{1}{2} X_l\epsilon q'\right)'.
$$

Note that $\frac{1}{2} X_l\epsilon = \frac{1}{2}(X_{lm} + X_{lf})$ is the average of his and her income effects. Equation (23) provides ten linear restrictions among the observable partial derivatives of (10) with respect to price $(X_p)$ and income $(X_l)$. The weights in these restrictions are taken from $G^{-1}$ whose columns are identified only up to a scalar multiple:

$^{12}$ This statement of Engel aggregation falls out of the derivation of the income effects. See equation (A. 7) in the Appendix.

$^{13}$ Post-multiplying the first line in (19) by $\epsilon 1/2$ allows one to solve for $b$ which is used in the substitution: $b = 1/2 (X_l - BDV_l)_t$.

$^{14}$ Using (18), the first line of (22) may be written as

$$
X_p = B(\lambda I + DV_p) - 1/2 (X_l - BDV_l)\epsilon q'.
$$

Solving for $B$ yields, $B = (X_p + 1/2 X_l\epsilon q')G^{-1}$, where $G = (\lambda I + DV_p) + 1/2 DV_l\epsilon q'$. Since $B = B'$, (23) follows.
\( G^{-1} = \hat{\beta} \left( X_p + \frac{1}{2} X_t q' \right)' \),

where \( \hat{\beta} \) is any diagonal matrix of order 5.

Finally, we note that the matrix \( B \) contained in equation (18) is symmetric, negative semidefinite and that Cournot aggregation is given by

\[ p'X_p = -q'. \]

These propositions are proved in the Appendix.

4.3. Comparison with the neoclassical comparative statics. Direct substitution of \( V_p = 0 \) and \( V_t = 0 \) shows that the uncompensated price effects (17), the compensated price effects (18) and the income effects (19), as well as the Nash generalizations of both the Slutsky equation (22) and substitution symmetry (23) collapse to their neoclassical counterparts when both threat points are independent of prices and nonwage incomes. These substitutions yield the neoclassical equations of comparative statics:

(17a) Uncompensated price effects:
\[ (V_p = 0) \]
\[ 
\begin{pmatrix}
X_p \\
-\hat{\beta}_p
\end{pmatrix}
= 
\begin{pmatrix}
\hat{\beta} B - b q' \\
\hat{\beta} b' - c^{-1} q'
\end{pmatrix},
\]

(18a) Compensated price effect:
\[ (V_p = 0) \]
\[ 
\begin{pmatrix}
X_p^* \\
-\hat{\beta}_p^ *
\end{pmatrix}
= 
\begin{pmatrix}
\hat{\beta} B \\
\hat{\beta} b'
\end{pmatrix},
\]

(19a) Income effects:
\[ (V_t = 0) \]
\[ 
\begin{pmatrix}
x_i \\
\hat{\beta}_t
\end{pmatrix}
= 
\begin{pmatrix}
b \\
c^{-1}
\end{pmatrix},
\]

(22a) Slutsky equation:
\[ (V_t = 0 \text{ and } V_p = 0) \]
\[ 
\begin{pmatrix}
x_i q' \\
\hat{\beta}_p + \hat{\beta}_t q'
\end{pmatrix}
= 
\begin{pmatrix}
S - x_i q' \\
\hat{\beta}_p + \hat{\beta}_t q'
\end{pmatrix},
\]

(23a) Substitution symmetry:
\[ (X_p + x_i q') = (X_p + x_i q'). \]

In (19a), (22a), and (23a) \( x_i \) stands for the common value, \( x_i = X_p = X_t = b \), and \( l_i \) for the common value, \( l_i = \hat{\beta}_m = \hat{\beta}_t = c^{-1} \). Likewise, in (22a) and (23a) \( S \) replaces \( X_p^* \), i.e., for \( V_p = 0 \) and \( V_t = 0 \), the neoclassical substitution matrix is

\[ \hat{\beta} B = \left( X_p + \frac{1}{2} X_t q' \right) = (X_p + x_i q') \equiv S. \]

Finally, note the Nash and neoclassical Cournot aggregation conditions (25), are identical (i.e., (25) is unaffected by \( V_p = 0 \)). In contrast, when \( V_t = 0 \), the Nash generalization of Engel aggregation (21), reduces to neoclassical Engel aggregation,

(21a)
\[ p'x_t = 1. \]

Hence, when \( V_p \) and \( V_t \) are null, the generalized Nash comparative statics and restrictions collapse to their neoclassical counterparts. This suggests testing the
following hypotheses:

(i) \( H_0: X_{lf} = X_{tl} \) versus \( H_A: X_{lm} \neq X_{tl} \),

or the effect on the demand for \( x_i \) of a change in male non-wage income is identical to the effect of a change in female non-wage income (19a).

(ii) \( H_0: (X_p + x_iq') = (X_p + x_iq')' \) versus \( H_A: \left( X_p + \frac{1}{2} X_{tf}q' \right) G^{-1} \)

\[ = (G^{-1})' \left( X_p + \frac{1}{2} X_{tf}q' \right)' , \]

or neoclassical substitution symmetry (23a) holds; and

(iii) \( H_0: X_p + x_iq' \) is negative semidefinite,

or the neoclassical substitution matrix is negative semidefinite.

A necessary condition for (iii) is

(iii') \( H_0: [X_p + x_iq']_i < 0 \) versus \( H_A: [X_p + x_iq']_i \geq 0 \),

or, neoclassical own substitution terms are negative.

Rejection of any of these hypotheses would be rejection of the neoclassical restrictions and would contradict the notion that the Nash demand model collapses to the neoclassical one. Of course, the aggregation conditions (25) and (21a) provide no further tests since any system of demand equations which identically satisfies the budget restraint will identically satisfy Cournot aggregation and our generalized Nash-Engel aggregation condition.

5. GRAPHICAL RELATIONSHIP BETWEEN THE NASH AND NEOCLASSICAL MODELS

5.1. Nash income effects. Figure 1 depicts IGPC's for his leisure, \( x_3 \), versus hers, \( x_4 \). For purposes of exposition, assume that in the relevant region the couple is selfish, i.e., \( AMRS_{34} < 0 \), so that, by (13), an increase in his income, \( I_m \), increases the family's valuation of his leisure in terms of hers, i.e., \( \frac{\partial FRS_{34}}{\partial I_m} > 0 \).

Suppose initially the budget restraint is \( KL \), the representative IGPC's are the (boldfaced) curves (labeled \( N' \)), and the optimal bundle is \( C \). An increase in \( I_m \) shifts the budget restraint out to \( WR \) and twists a representative IGPC to one such as the dotted curve (labeled \( N'' \)); consequently, the final optimal bundle is \( F \). Comparison of (19) and (19a) shows that the total Nash income effect, \( CF \) in Figure 1, can be resolved into a neoclassical income effect, \( CE \), and an effect, \( EF \), due to the tilt in the family of IGPC's:

<table>
<thead>
<tr>
<th>total income effect</th>
<th>=</th>
<th>neoclassical income effect</th>
<th>+</th>
<th>IGP income tilt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1</td>
<td>( CF )</td>
<td>( CE )</td>
<td>+</td>
<td>( EF )</td>
</tr>
<tr>
<td>Equation (19)</td>
<td>( [X_{lm}, X_{tl}] )</td>
<td>( [b, b] )</td>
<td>+</td>
<td>( BDV_t )</td>
</tr>
<tr>
<td>Equation (19a)</td>
<td>( x_t = b )</td>
<td>( (V_t = 0) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^{15}\) Manser and Brown [1978] suggest similar tests.
5.2. *Nash uncompensated and income-compensated price effects.* Continue with the assumption of selfishness in the relevant region so that, by (13), an increase in his wage, $p_3$, increases the family's valuation of his leisure in terms of hers, i.e., $\frac{\partial FRS_{34}}{\partial p_3} > 0$. Assume initially that the budget restraint is $QR$, the family of IGPC's is represented by the curve labeled $N=a$, and the optimal bundle is $A$. An increase in his wage, $p_3$, rotates the budget line to $WR$ and twists the family of IGPC's to the family of (boldfaced) curves (labeled $N'$); consequently the final optimal bundle is at $E$. Thus $AE$ is the Nash uncompensated price effect.

![Diagram of Nash uncompensated and income-compensated effects of an increase in $p_3$ on the optimal quantities of $x_3$ and $x_4$.]

For this change in $p_3$ the corresponding income-compensated price effect is $AD$, where $D$ is the tangency of the new IGPC with value $N'=a$ to a budget line reflecting the new price ratio. This Nash income-compensated price effect may be resolved into three components as follows:

\[
\text{income compensated price effect} = \text{neoclassical substitution effect} + \text{IGP price tilt effect} + \text{relabeling effect}
\]

<table>
<thead>
<tr>
<th>Figure 1</th>
<th>$AD$</th>
<th>$AH$</th>
<th>$HC$</th>
<th>$CD$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation (18)</td>
<td>$X_p^*$</td>
<td>$\dot{\lambda}B$</td>
<td>$BDV_p$</td>
<td>$\dot{\lambda}^{-1}b'V_p$</td>
</tr>
<tr>
<td>Equation (18a)</td>
<td>$S$</td>
<td>$\dot{\lambda}B$</td>
<td>$(V_p=0)$</td>
<td></td>
</tr>
</tbody>
</table>
Comparing (18) with (18a) and Figure 1, it is easily seen that the change in the optimal bundle \( AH \) can be identified with the term \( \lambda B \) which is the neoclassical substitution effect. Furthermore, let \( C \) be the tangency of the new (boldfaced) IGPC through \( H \) tangent to a (dashed) budget line reflecting the new price ratio. Then we can identify the term \( BDV_p \) with the change in the optimal bundle, \( HC \), which is due to the tilt in the IGPC’s associated with the increase in \( p_3 \).

Finally, the residual component of the compensated price effect is \( CD \) which corresponds to \( \lambda^{-1}bg'V_p \) in (18) and is called the “relabelling effect.” By (16), the new \( N' = a \) curve lies everywhere to the northeast of the old \( N = a \) curve and does not intersect it. Since a monotone transformation (or relabelling) of IGPC’s cannot affect choice, this relabelling effect will drop out of the Nash generalization of the Slutsky equation.

5.3. Nash generalization of the Slutsky equation. Combining the above effects gives us the Nash generalization of the Slutsky equation which resolves the total effect of a price change into appropriately adjusted substitution and income effects:

\[
\begin{align*}
\text{total price effect} & = (\text{compensated price effect} - \text{relabelling}) + (\text{income effect} - \text{IGP income tilt}) \\
\text{Figure 1} & \quad AE = (AD - CD) + (CF - EF) \\
& \quad AE = AC + CE \\
\text{Equation (22)} & \quad X_{p} = (X_{p}^\star - \lambda^{-1}bg'V_p) - \frac{1}{2}(Y_f - BDV_f)q' \\
\text{Equation (22a)} & \quad X_{p} = S - x_iq'
\end{align*}
\]

Notice that the substitution term is adjusted by subtracting off the relabelling effect \( (CD, or \lambda^{-1}bg'V_p) \). Similarly, the income term is adjusted by subtracting off the effect on the optimal bundle of the income-compensation (i.e., subtracting off \( BDV_f \) or deleting the move \( EF \)). Consequently, in Figure 1, if she rather than he had received the income compensation associated with the change in \( p_3 \), the total income effect would have been, say, \( CG \), but the adjusted income effect would have remained unchanged, \( CE \). Thus, both the adjusted income effect and the Nash generalization of the Slutsky equation are invariant with respect to the recipient of the (hypothetical) income compensation. This makes sense: for the total price effect, \( X_{p} \), the net change in the IGPC’s should reflect only the price changes and not a change in \( I_m \) or \( I_f \).

6. CONCLUDING REMARKS

This paper presents a theory of demand associated with a Nash bargaining model of decision-making for a two-person household. The Nash objective

\[\frac{\partial N_i}{\partial p_h} = 0 \text{ for all } i \text{ and } h \text{ then, by (13), } \frac{\partial F_{i,j}}{\partial p_h} = 0 \text{ for all } i, j \text{ and } h. \text{ Thus, } D_{V_p} = 0 \implies \frac{\partial F_{i,j}}{\partial p_h} = 0.\]
function depends upon individual utilities. It also depends upon the maximum value of (indirect) utility each person can obtain outside of the household and it is hence dependent upon prices and nonwage incomes. Consequently the nonwage income of each spouse appears as an independent variable in each demand equation. The Nash demand system retains the neoclassical properties of homogeneity and Cournot aggregation. It also exhibits what we call the Nash generalizations of Engel aggregation, of the Slutsky equation and of substitution symmetry. These three generalizations lead naturally to tests of the hypotheses that the Nash demand restrictions collapse to the corresponding neoclassical ones.

In a companion paper (Horney and McElroy [1980]), we specified a three commodity (male leisure, female leisure, and a Hicksian composite commodity) Nash linear expenditure system and estimated it for the 1967 National Longitudinal Survey of Mature Women. On the whole, the results were reasonable and there were indications that the Nash linear expenditure system does not collapse to the neoclassical one. Brown and Manser [1978], using a somewhat different Nash bargaining specification and a Rotterdam-type demand system, also found evidence to reject the hypothesis that a Nash demand system collapses to a neoclassical one. These studies point to the empirical enrichment available from bargaining models as opposed to individual decision models. The empirical payoff includes both richer functional forms for demand systems and additional explanatory variables incorporated via the threat points.

This richer analytical framework has many potential applications. For example, the Nash model provides for the first time an analytical framework for examining the effect of the "marriage tax" on the joint labor supply of husbands and wives: her current, married after-tax marginal wage rate is an argument of the family full income constraint; whereas her hypothetical unmarried after-tax marginal wage rate is an argument of her threat point. This is only one of a general class of applications where, due to taxes and transfers, the prices and nonwage incomes faced by an individual differ according to his or her marital status.

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APPENDIX

In this Appendix we derive the comparative static equations of the bargaining model — equations (17), (18), and (19) of the text.

Uncompensated price effects. Partial differentiation of each equation in (8) and (9) with respect to each price, holding other prices and nonwage income constant yields the matrix equation,
(A.1) \[
\begin{bmatrix}
J & p \\
p' & 0
\end{bmatrix}
\begin{bmatrix}
X_p \\
- \lambda_p'
\end{bmatrix}
= \begin{bmatrix}
\lambda I_5 + DV_p \\
- q'
\end{bmatrix},
\]

where

\[
J = [N_{ij}] = [U_i u_{ij} + u_{ij}U_j + U_i(U_j - V_j) + U_j(U - V_j)].
\]

The partitioned inverse of the leftmost matrix in (A.1) is

(A.2) \[
\begin{bmatrix}
J & p \\
p' & 0
\end{bmatrix}^{-1} = \begin{bmatrix}
J^{-1} - (p' J^{-1} p)^{-1} J^{-1} p p' J^{-1} J^{-1} p (p' J^{-1} p)^{-1} \\
(p' J^{-1} p)^{-1} p' J^{-1} \\
- (p' J^{-1} p)^{-1}
\end{bmatrix}
\begin{bmatrix}
B & b \\
b' & -c^{-1}
\end{bmatrix}
\]

Premultiplication of (A.1) by (A.2) yields the matrix of uncompensated price effects on the optimal \( x \) and \( \lambda \).

(A.3) \[
\begin{bmatrix}
X_p \\
- \lambda_p'
\end{bmatrix}
= \begin{bmatrix}
B & b \\
b' & -c^{-1}
\end{bmatrix}
\begin{bmatrix}
\lambda I_5 + DV_p \\
- q'
\end{bmatrix}
= \begin{bmatrix}
B(\lambda I_5 + DV_p) - bq' \\
b'(\lambda I_5 + DV_p) + c^{-1}q'
\end{bmatrix},
\]

which is equation (17) in the text. Finally, note that (A.1) contains the Cournot aggregation condition,

(A.3) \[
p' X_p = - q',
\]

which is equation (25) in the text.

Compensated price effects. For a fixed value of \( N \) in (7), partial differentiation of (7) and (8) with respect to each price, holding all other prices constant, (and substituting from (8)) yields

(A.5) \[
\begin{bmatrix}
J & p \\
p' & 0
\end{bmatrix}
\begin{bmatrix}
X_p^* \\
- \lambda_p^{* '}
\end{bmatrix}
= \begin{bmatrix}
\lambda I_5 + DV_p \\
- c^{-1}g' V_p
\end{bmatrix},
\]

Premultiplication of (A.5) by (A.2) yields the matrix of compensated price effects,

(A.6) \[
\begin{bmatrix}
X_p^* \\
- \lambda_p^{* '}
\end{bmatrix}
= \begin{bmatrix}
B & b \\
b' & -c^{-1}
\end{bmatrix}
\begin{bmatrix}
\lambda I_5 + DV_p \\
- c^{-1}g' V_p
\end{bmatrix}
= \begin{bmatrix}
B(\lambda I_5 + DV_p) + c^{-1} g' V_p \\
b'(\lambda I_5 + DV_p) - (c\lambda)^{-1}g' V_p
\end{bmatrix},
\]

which is equation (18) in the text.

By Young's theorem, \( J \) is symmetric and therefore \( B \) is symmetric. From the first line of (A.6) we can see that \( \lambda B \) is negative semi-definite. In the neoclassical case, \( V_p \) is null and the substitution matrix \( S = \lambda B \). Then, since the \( J \) matrix is
just a matrix of second partials of a quasiconcave objective function the standard proof of the negative semidefiniteness of $S$ holds. Therefore $J_B$ is negative semidefinite whether or not $V_p$ is null.

**Income effects.** Finally, holding prices constant, partial differentiation of (8) and (9) with respect to $I_m$ and $I_f$ yields

\[
(A.7) \begin{bmatrix} J & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} X_l \\ -\lambda_l \end{bmatrix} = \begin{bmatrix} D V_l \\ \epsilon' \end{bmatrix}.
\]

Premultiplying (A.7) by (A.2), yields the matrix of Nash income effects,

\[
(A.8) \begin{bmatrix} X_l \\ -\lambda_l \end{bmatrix} = \begin{bmatrix} B & h \\ b' - c^{-1} & \epsilon' \end{bmatrix} \begin{bmatrix} D V_l \\ \epsilon' \end{bmatrix} = \begin{bmatrix} B D V_l + h \epsilon' \\ b' D V_l - c^{-1} \epsilon' \end{bmatrix},
\]

which is equation (19) in the text.

**REFERENCES**


